An alternative approach to exact wave functions for time-dependent coupled oscillator model of a charge in variable magnetic field

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Abstract

The quantum states of time-dependent coupled oscillator model for charged particles subjected to variable magnetic field are investigated using the invariant operator methods. To do this, we have taken advantage of an alternative method, so-called unitary transformation approach, available in the framework of quantum mechanics, as well as a generalized canonical transformation method in the classical regime. The transformed quantum Hamiltonian is obtained using suitable unitary operators and is represented in terms of two independent harmonic oscillators which have the same frequencies as that of the classically transformed one. Starting from the wave functions in the transformed system, we have derived the full wave functions in the original system with the help of the unitary operators. One can easily take a complete description of how the charged particle behaves under the given Hamiltonian by taking advantage of these analytical wave functions.

Key Words: *Time-dependent systems; coupled oscillator; unitary transformation Schrödinger equation.*

1. INTRODUCTION

Since the pioneering works of Lewis[1–3], the investigation of the quantum behavior for time-dependent harmonic oscillator has attracted considerable interest in the literature because it offers exactly soluble models for dynamical systems in different areas of physics. There are diverse kind of time-dependent harmonic oscillators such as Caldirola-Kanai oscillator[4, 5], parametric oscillator[6], and harmonic oscillator with a strongly pulsating mass[7], The two-dimensional time-dependent harmonic oscillator also became an hot topic in both classical and quantum mechanics. The higher dimensional harmonic oscillator has played an important role in, for examples, nuclear shell structure and models of quark confinement. According to the progress of research for these systems, a relevant question has been naturally raised: what would happens if two-dimensional harmonic oscillator is constructed by the coupling of external additive potentials? The investigation for this issue was initiated by Kim et al.[8–13] about thirty years ago. They considered two harmonic oscillators that are mutually-coupled so that the resulting potential becomes $V(x_1, x_2) = \frac{1}{2}(c_1x_1^2 + c_2x_2^2 + c_3x_1x_2)$. They determined the corresponding density matrix rigorously in order to establish the Wigner function and some of other useful functions in physics. There are plenty of physical systems and models described by coupled harmonic oscillators, such as the Lee model in quantum field theory [14], the Bogoliubov transformation model of superconductivity [15], two-mode squeezed light [16], the covariant harmonic oscillator model for theparton picture [17], and some models in molecular physics [18]. One of the main focuses of research fulfilled in connection with time-dependent coupled oscillators is some specific problems of time-dependent coupled electric circuit whose closed-form solutions are now well known[19–22]. And further, the author of Ref. [23] have investigated the propagator for a certain class of time-dependent coupled and driven harmonic oscillators with time-varying frequencies and masses using path integral methods.

Above all, a charged particle in a strong uniform magnetic field is a typical example of natural non-commutative system [24], which provides a good starting point when we are going to discuss the quantum Hall effect [25]. The external magnetic field is indeed an important factor that affects to the motion of a charged particle in various physical systems. When a time-dependent

magnetic field is exerted on an electron, it is impossible to reduce the system to be a one-dimensional problem. Instead, it can be modeled by two-dimensional time-dependent harmonic oscillator due to the existence of variable magnetic field $B(t)$. Theoretical and experimental researches have been carried out extensively on the quantum properties of this system in the past several decades due to its importance not only in condensed matter physics but also in plasma physics[26–36]. Especially, the study of charged particle motion driven by external magnetic field is crucial in investigating magnetic confinement devices for fusion plasmas (whose subtopics are tokamaks, mirror machines, bumpy tori, and stellarators) and for space and astrophysical plasmas (whose subtopics are magnetospheric plasmas of earth and other planets like pulsar)[36]. The wave functions of a free electron with time-dependent effective mass, in the presence of a variable magnetic field, are derived in both the Landau and symmetric gauges [37]. The propagators of a charged particle subjected to a time-dependent magnetic field, which propagate the wave functions in the spacetime, are derived using the linear and the quadratic invariants [38].

An interesting problem that would be worth to be dealt with is the charged particle system that is described by the Hamiltonian which involves the static coupling xy and dynamic coupling term $p_x p_y$ under the presence of magnetic field. This system may exhibit novel features owing to the existence of the coupling terms and can be regarded as the generalization of the Hamiltonian model given in Refs. [31] and [39]. Our intention in this paper is to calculate the exact wave function for time-dependent coupled oscillators in a variable magnetic field within the framework of the invariant methods. The calculation is based on the use of the generalized time-dependent canonical transformations and an alternative time-dependent unitary transformation.

The present paper is organized as follows. Our problem is formulated in section 2 through a general time-dependent Hamiltonian describing the motion of a complicate charged particle system. Some remarks necessary in dealing with our task will also be presented. In section 3, we show how to simplify the problem associated with the complicate Hamiltonian of our system using the canonical transformation method. As an alternative approach, unitary transformation

is also applied, in section 4, in order to transform our complicate Hamiltonian to that of a more simplified harmonic oscillator. The quantum solution of the system will be investigated in section 5 using the invariant methods on the basis of the results obtained in section 4. The concluding remarks are given in the last section.

2. FORMULATING THE PROBLEM

Let us formulate our problem by introducing a generalized Hamiltonian describing the motion of a charged particle that have time-dependent effective mass $m(t)$ in the presence of a variable magnetic field. The effective mass of charged particles, such as electrons or holes in any system, may modified through their interaction with surroundings or various excitations like energy[40], stress[41], temperature[42], and pressure[43]. It is therefore natural to think that the effective mass varies with time according to the change of the environments. Moreover, if we vary the external magnetic field randomly in the heterojunctions and solid solutions, the effective mass of an electron also varies in a random fashion in response to the fluctuation of the composition in the system[44].

We consider the electromagnetic potential in the symmetric gauge such that $\overrightarrow{A}\left(\frac{-B(t)}{2}y, \frac{B(t)}{2}x, 0\right)$ where x and y are the position operators. Then, for the dynamical system of our interest, the Hamiltonian has the form

$$H(x,y,t) = \frac{1}{2m(t)}\left(\pi_x^2 + \pi_y^2\right) + \frac{1}{2}m(t)\varpi^2(t)\left(x^2+y^2\right) + a(t)xy + b(t)p_xp_y, \tag{1}$$

where $\pi_x = p_x - \frac{eB(t)y}{2}$ and $\pi_y = p_y + \frac{eB(t)x}{2}$ while p_x and p_y are conjugate momentum operators that are given by $p_x = -i\hbar\partial/\partial x$ and $p_y = -i\hbar\partial/\partial y$. $\varpi(t)$ is the oscillating frequency that is an arbitrary function of time. To generalize the problem, we suppose that the other parameters, $m(t)$, $a(t)$ and $b(t)$, are also arbitrary time functions.

In terms of p_x and p_y, the Hamiltonian (1) can be rewritten as

$$H(x,y,t) = \frac{1}{2m(t)}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m(t)\omega^2(t)\left(x^2+y^2\right) + a(t)xy + b(t)p_xp_y + \frac{\varpi_c(t)}{2}L_z. \tag{2}$$

Here, $\omega(t)$ is a modulation frequency which takes the form $\omega^2(t) = \varpi^2(t) + \frac{\varpi_c^2(t)}{4}$ where $\varpi_c(t) =$

$\frac{eB(t)}{M(t)}$ is the Larmor frequency, and $L_z = xp_y - p_x y = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$ is the canonical angular momentum in the axial z direction. The study of quantum features for this system may be a fascinating task in its own right both from a physical and a mathematical point of view. We will show how to convert this Hamiltonian to a simple form in the following two sections by means of the canonical transformation and unitary transformation, respectively. These procedures may enables us to derive the quantum solutions of the system.

3. CANONICAL TRANSFORMATION

The method of time-dependent canonical transformation is in fact very powerful in investigating the mechanical behavior of dynamical systems. We can convert a given Hamiltonian into any desired one through this method. In order to cast the Hamiltonian of our problem into a more soluble form, we take the advantage of the time-dependent canonical transformation $(x, y, p_x, p_y) \rightarrow \ (X, Y, P_X, P_Y)$ defined as.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}, \tag{3}$$

$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} \cos\phi(t) & \sin\phi(t) \\ -\sin\phi(t) & \cos\phi(t) \end{pmatrix} \begin{pmatrix} P_X \\ P_Y \end{pmatrix}, \tag{4}$$

where

$$\phi(t) = -\frac{1}{2}\int \varpi_c(t)dt. \tag{5}$$

From the fundamentals of classical mechanics, we have [45]

$$p_x = \frac{\partial}{\partial x}F_1(x.y, P_X, P_Y, t) \quad , X = \frac{\partial}{\partial P_X}F_1(x.y, P_X, P_Y, t) \ , \tag{6}$$

$$p_y = \frac{\partial}{\partial y}F_1(x.y, P_X, P_Y, t) \quad , Y = \frac{\partial}{\partial P_Y}F_1(x.y, P_X, P_Y, t), \tag{7}$$

and

$$(P_X\dot{X} + P_Y\dot{Y} - H(X, Y, t) = p_x\dot{x} + p_y\dot{y} - H(x, y, t) + \frac{\partial F_1}{\partial t}, \tag{8}$$

where F_1 is the generating functions responsible for the transformation. Through these relations, F_1 is easily found to be

$$F_1\left(x.y, P_X, P_Y, t\right) = \left(P_X \cos\phi + P_Y \sin\phi\right) x + \left(-P_X \sin\phi + P_Y \cos\phi\right) y, \tag{9}$$

$$\frac{\partial F_1}{\partial t} = -\dot{\phi}(t) L_z = -\frac{\varpi_c(t)}{2} L_z. \tag{10}$$

In terms of the new conjugate variables $(X, Y, P_X, P_Y,)$, the Hamiltonian (2) becomes

$$\begin{aligned} H(X,Y,t) \;=\; & \frac{1}{2m_-(t)} P_X^2 + \frac{1}{2m_+(t)} P_Y^2 + \frac{1}{2} m_-(t)\omega_-^2(t) X^2 + \frac{1}{2} m_+(t)\omega_+^2(t) Y^2 \\ & + a_1(t) XY + b_1(t) P_X P_Y, \end{aligned} \tag{11}$$

where

$$\frac{1}{m_-(t)} = \frac{1}{m(t)} - 2b(t)\sin\phi\cos\phi, \tag{12}$$

$$\frac{1}{m_+(t)} = \frac{1}{m(t)} + 2b(t)\sin\phi\cos\phi, \tag{13}$$

$$\omega_-(t) = \left(\frac{m(t)\omega^2(t) - 2a(t)\sin\phi\cos\phi}{m_-(t)}\right)^{1/2}, \tag{14}$$

$$\omega_+(t) = \left(\frac{m(t)\omega^2(t) + 2a(t)\sin\phi\cos\phi}{m_+(t)}\right)^{1/2}, \tag{15}$$

$$a_1(t) = a(t)\left(\cos^2\phi - \sin^2\phi\right), \tag{16}$$

$$b_1(t) = b(t)\left(\cos^2\phi - \sin^2\phi\right). \tag{17}$$

To eliminate the dynamical term $P_X P_Y$, we take the second canonical transformation by recasting the canonical variables (X, Y, P_X, P_Y) in terms of new variables (q_1, q_2, p_1, p_2):

$$X = \left(\frac{m_+(t)}{m_-(t)}\right)^{1/4} \left(\frac{q_1 + q_2}{\sqrt{2}}\right) \quad , \quad Y = \left(\frac{m_-(t)}{m_+(t)}\right)^{1/4} \left(\frac{-q_1 + q_2}{\sqrt{2}}\right) \;, \tag{18}$$

$$P_X = \left(\frac{m_-(t)}{m_+(t)}\right)^{1/4} \left(\frac{p_1 + p_2}{\sqrt{2}}\right) \quad , \quad P_Y = \left(\frac{m_+(t)}{m_-(t)}\right)^{1/4} \left(\frac{-p_1 + p_2}{\sqrt{2}}\right). \tag{19}$$

The canonical transformation based on Eqs. (18) and (19) enables us to transform $H(X, Y, t)$ into $H(q_1, q_2, t)$. Thus, straightforwardly, we have

$$H(q_1, q_2, t) = \frac{1}{2m_1(t)} p_1^2 + \frac{1}{2m_2(t)} p_2^2 + \frac{1}{2} m_1(t)\omega_1^2(t) q_1^2 + \frac{1}{2} m_2(t)\omega_2^2(t) q_2^2 + c(t) q_1 q_2, \tag{20}$$

where

$$\frac{1}{m_1(t)} = \frac{1}{\sqrt{m_+(t)m_-(t)}} - b_1(t), \tag{21}$$

$$\frac{1}{m_2(t)} = \frac{1}{\sqrt{m_+(t)m_-(t)}} + b_1(t), \tag{22}$$

$$\omega_1(t) = \left(\frac{\frac{1}{2}\sqrt{m_+(t)m_-(t)}\left[\left(\omega_-^2(t) + \omega_+^2(t)\right)\right] - a_1(t)}{m_1(t)} \right)^{1/2}, \tag{23}$$

$$\omega_2(t) = \left(\frac{\frac{1}{2}\sqrt{m_+(t)m_-(t)}\left[\left(\omega_-^2(t) + \omega_+^2(t)\right)\right] + a_1(t)}{m_2(t)} \right)^{1/2}, \tag{24}$$

$$c(t) = \frac{1}{2}\sqrt{m_+(t)m_-(t)}\left(\omega_-^2(t) - \omega_+^2(t)\right). \tag{25}$$

Now, to remove the static coupling term q_1q_2, we take another canonical transformation by introducing the variables $(Q_i,\, P_i)$ where $i = 1, 2$, such that [20, 21, 23]

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{m_1(t)}}\cos\frac{\theta(t)}{2} & \frac{1}{\sqrt{m_1(t)}}\sin\frac{\theta(t)}{2} \\ -\frac{1}{\sqrt{m_2(t)}}\sin\frac{\theta(t)}{2} & \frac{1}{\sqrt{m_2(t)}}\cos\frac{\theta(t)}{2} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \tag{26}$$

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \sqrt{m_1(t)}\cos\frac{\theta(t)}{2} & \sqrt{m_1(t)}\sin\frac{\theta(t)}{2} \\ -\sqrt{m_2(t)}\sin\frac{\theta(t)}{2} & \sqrt{m_2(t)}\cos\frac{\theta(t)}{2} \end{pmatrix} \begin{pmatrix} P_1 \\ P_2 \end{pmatrix} - \begin{pmatrix} \frac{\dot{m}_1(t)}{2} & 0 \\ 0 & \frac{\dot{m}_2(t)}{2} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \tag{27}$$

where $\theta(t)$ is an arbitrary phase which shall be appropriately determined afterwards. Equations (26) and (27) do not always represent canonical transformation[45] between the variables (q_i, p_i) and variables (Q_i, P_i). If (Q_i, P_i) are canonical coordinates, there should exist a new Hamiltonian which is determined only by the Hamiltonian given in (20) and the linear transformation of (26) and (27).

The variables (q_i, p_i) and (Q_i, P_i) in the two representations must satisfy the following relation[45]

$$\sum_{i=1}^{2} P_i\dot{Q}_i - H_Q = \sum_{i=1}^{2} p_i\dot{q}_i - H_q + \frac{\partial F}{\partial t}, \tag{28}$$

provided that the transformation is canonical, where F is called a generating function, possibly a time-dependent function in phase space. From the equations known in classical mechanics

$$p_i = \frac{\partial}{\partial q_i}F\left(q_1, q_2, P_1, P_2, t\right) \quad , \; Q_i = \frac{\partial}{\partial P_i}F\left(q_1, q_2, P_1, P_2, t\right) \qquad i = 1, 2, \tag{29}$$

the generating function responsible for the transformation is found to be

$$\begin{aligned} F\left(q_1, q_2, P_1, P_2, t\right) &= \sqrt{m_1(t)}\left(P_1 \cos\frac{\theta(t)}{2} + P_2 \sin\frac{\theta(t)}{2}\right) q_1 \\ &\quad + \sqrt{m_2(t)}\left(-P_1 \sin\frac{\theta(t)}{2} + P_2 \cos\frac{\theta(t)}{2}\right) q_2 \\ &\quad + \frac{1}{2}\left(-\frac{1}{2}\dot{m}_1(t) q_1^2 - \frac{1}{2}\dot{m}_2(t) q_2^2\right). \end{aligned} \tag{30}$$

In terms of the new conjugate variables (Q_i, P_i) the Hamiltonian of the system can be rewritten as

$$\begin{aligned} H_Q(Q_1, Q_2, t) &= \frac{1}{2}\left(P_1^2 + P_1^2\right) + \frac{1}{2}\Omega_1^2(t) Q_1^2 + \frac{1}{2}\Omega_2^2(t) Q_2^2 \\ &\quad + \frac{\dot{\theta}(t)}{2}\left[P_1 Q_2 - P_2 Q_1\right] + \delta(t) Q_1 Q_2, \end{aligned} \tag{31}$$

where the time-dependent coefficients $\Omega_1(t), \Omega_2(t)$ and $\delta(t)$ are given by

$$\Omega_1(t) = \left(\tilde{\omega}_1^2(t)\cos^2\frac{\theta(t)}{2} + \tilde{\omega}_2^2(t)\sin^2\frac{\theta(t)}{2} - \frac{c(t)\sin\theta(t)}{\sqrt{m_1(t) m_2(t)}}\right)^{1/2}, \tag{32}$$

$$\Omega_2(t) = \left(\tilde{\omega}_1^2(t)\sin^2\frac{\theta(t)}{2} + \tilde{\omega}_2^2(t)\cos^2\frac{\theta(t)}{2} + \frac{c(t)\sin\theta(t)}{\sqrt{m_1(t) m_2(t)}}\right)^{1/2}, \tag{33}$$

$$\delta(t) = \frac{1}{2}\left(\tilde{\omega}_1^2(t) - \tilde{\omega}_2^2(t)\right)\sin\theta(t) + \frac{c(t)\cos\theta(t)}{\sqrt{m_1(t) m_2(t)}}, \tag{34}$$

with

$$\tilde{\omega}_1^2(t) = \omega_1^2(t) + \frac{1}{4}\left(\frac{\dot{m}_1^2(t)}{m_1^2(t)} - 2\frac{\ddot{m}_1(t)}{m_1(t)}\right), \tag{35}$$

$$\tilde{\omega}_2^2(t) = \omega_2^2(t) + \frac{1}{4}\left(\frac{\dot{m}_2^2(t)}{m_2^2(t)} - 2\frac{\ddot{m}_2(t)}{m_2(t)}\right). \tag{36}$$

If we choose $\theta(t) = \text{Const.}$, the term involving $P_1 Q_2$ and $P_2 Q_1$ in Eq. (31) disappears and, consequently, we have

$$H_Q(Q_1, Q_2, t) = \frac{1}{2}\left(P_1^2 + P_1^2\right) + \frac{1}{2}\Omega_1^2(t) Q_1^2 + \frac{1}{2}\Omega_2^2(t) Q_2^2 + \delta(t) Q_1 Q_2. \tag{37}$$

It is notable that, with the above canonical transformation, the coupling $\delta(t)$ is a function of the parameters of the original system. Evidently, the separation of variables in equation (37) requires that $\delta(t) = 0$, i.e.

$$c(t) = \frac{1}{2}\left(\tilde{\omega}_2^2(t) - \tilde{\omega}_1^2(t)\right)\sqrt{m_1(t)m_2(t)}\tan\theta, \tag{38}$$

and consequently

$$\tan\theta = \frac{\sqrt{m_+(t)m_-(t)}\left(\omega_-^2(t) - \omega_+^2(t)\right)}{\sqrt{m_1(t)m_2(t)}\left(\tilde{\omega}_2^2(t) - \tilde{\omega}_1^2(t)\right)}. \tag{39}$$

Under this condition, the Hamiltonian of Eq. (37) reduces to

$$H_Q(Q_1, Q_2, t) = \frac{1}{2}\left(P_1^2 + P_1^2\right) + \frac{1}{2}\Omega_1^2(t)Q_1^2 + \frac{1}{2}\Omega_2^2(t)Q_2^2. \tag{40}$$

This is the sum of two individual Hamiltonians corresponding to the harmonic oscillators having the time-dependent frequencies $\Omega_1(t)$ and $\Omega_2(t)$, respectively, and having masses that are equal to unity.

4. UNITARY TRANSFORMATIONS

The unitary transformations in quantum mechanics is analogous to the canonical transformations in classical mechanics. In this section, the relationship between the two transformations will be demonstrated and we confirm how to obtain the quantum-mechanical Hamiltonian from the classical one. With the consideration of quantum physics, we replace the canonical variables (x, y) by quantum operators $(\hat{x}, \hat{y})$, so that the corresponding Hamiltonian is given by

$$\hat{H}(\hat{x}, \hat{y}, t) = \frac{1}{2m(t)}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}m(t)\omega^2(t)\left(\hat{x}^2 + \hat{y}^2\right) + a(t)\hat{x}\hat{y} + b(t)\hat{p}_x\hat{p}_y + \frac{\varpi_c(t)}{2}\hat{L}_z. \tag{41}$$

In fact, it is not difficult to show the commutation relations $[\hat{L}_z\ ,\ \hat{x}^2+\hat{y}^2] = 0$ and $[\hat{L}_z\ ,\ \hat{p}_x^2+\hat{p}_y^2] = 0$. We can also check the non-commutability of $\hat{L}_z$ with some other variables: $[\hat{L}_z\ ,\ \hat{p}_x\hat{p}_y] \neq 0$, and $[\hat{L}_z\ ,\ \hat{x}\hat{y}] \neq 0$ and consequently $[\hat{L}_z\ ,\ \hat{H}] \neq 0$. Considering this fact, we are unable to decompose the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(x, y, t) = \hat{H}(\hat{x}, \hat{y}, t)\Psi(x, y, t), \tag{42}$$

when we would like to simplify it, because the angular momentum operator $\hat{L}_z$ and the Hamiltonian $\hat{H}(\hat{x},\hat{y},t)$ does not have the same eigenstates. To overcome this difficult situation, we transform the Hamiltonian (41) to a simple form by means of appropriate unitary operators. In the first step, we perform the unitary transformation

$$\Psi(x,y,t) = \hat{U}(t)\psi((x,y,t), \tag{43}$$

where $U(t)$ is a unitary operator of the form

$$\hat{U}(t) = \exp\left(-\frac{i\hat{L}_z}{2\hbar}\int \varpi_c(t)dt\right). \tag{44}$$

Under this transformation, the Schrödinger equation of original systems (42) is mapped into

$$i\hbar\frac{\partial}{\partial t}\psi(x,y,t) = \hat{H}_1(\hat{x},\hat{y},t)\psi(x,y,t), \tag{45}$$

where the new Hamiltonian $\hat{H}_1(\hat{x},\hat{y},t)$ has the form

$$\hat{H}_1(\hat{x},\hat{y},t) = \frac{1}{2m_-(t)}\hat{p}_x^2 + \frac{1}{2m_+(t)}\hat{p}_y^2 + \frac{1}{2}m_-(t)\omega_-^2(t)\hat{x}^2 + \frac{1}{2}m_+(t)\omega_+^2(t)\hat{y}^2 + a_1(t)\hat{x}\hat{y} + b_1(t)\hat{p}_x\hat{p}_y. \tag{46}$$

Note that the term involving $\hat{L}_z$ disappeared in the above equation, This means that the magnetic field is removed when it is viewed from an appropriate rotating frame $\phi(t) = -\frac{1}{2}\int \varpi_c(t)dt$.

To simplify the Hamiltonian (46), we use two-step unitary transformation approach. As a first step, we take the following unitary transformation

$$\psi(x,y,t) = \hat{\Lambda}(t)\varphi(x,y,t), \tag{47}$$

where

$$\hat{\Lambda}(t) = \hat{\Lambda}_1(t)\hat{\Lambda}_2(t), \tag{48}$$

and $\hat{\Lambda}_1(t)$ and $\hat{\Lambda}_2(t)$ are given by

$$\hat{\Lambda}_1(t) = \exp\left[\frac{i}{2\hbar}(\hat{p}_x\hat{x} + \hat{x}\hat{p}_x)\ln\left(\frac{m_-(t)}{m_+(t)}\right)^{1/4}\right]\exp\left[\frac{i}{2\hbar}(\hat{p}_y\hat{y} + \hat{y}\hat{p}_y)\ln\left(\frac{m_+(t)}{m_-(t)}\right)^{1/4}\right], \tag{49}$$

$$\hat{\Lambda}_2(t) = \exp\left[-\frac{i}{\hbar}\frac{\pi}{4}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\right]. \tag{50}$$

Then, we can transform the Hamiltonian (46) using the formula

$$\hat{H}_2(\hat{x}, \hat{y}, t) = \hat{\Lambda}^{-1}(t)\hat{H}_1(\hat{x}, \hat{y}, t)\hat{\Lambda}(t) - i\hbar\hat{\Lambda}^{-1}(t)\frac{\partial}{\partial t}\hat{\Lambda}(t). \tag{51}$$

After some algebra, we get

$$\hat{H}_2(\hat{x}, \hat{y}, t) = \frac{1}{2m_1(t)}\hat{p}_x^2 + \frac{1}{2m_2(t)}\hat{p}_y^2 + \frac{1}{2}m_1(t)\omega_1^2(t)\hat{x}^2 + \frac{1}{2}m_2(t)\omega_2^2(t)\hat{y}^2 + c(t)\hat{x}\hat{y}. \tag{52}$$

In the next transformation we will eliminate the coupled static terms $\hat{x}\hat{y}$. To do this we consider the unitary transformation

$$\varphi(x, y, t) = \hat{V}(t)\chi(x, y, t). \tag{53}$$

Here, $\hat{V}(t)$ is a time-dependent unitary operator of the form

$$\hat{V}(t) = \hat{V}_1(t)\hat{V}_2(t)\hat{V}_3(t), \tag{54}$$

where

$$\hat{V}_1(t) = \exp\left[\frac{i}{2\hbar}(\hat{p}_x\hat{x} + \hat{x}\hat{p}_x)\ln\sqrt{m_1(t)}\right]\exp\left[\frac{i}{2\hbar}(\hat{p}_y\hat{y} + \hat{y}\hat{p}_y)\ln\sqrt{m_2(t)}\right], \tag{55}$$

$$\hat{V}_2(t) = \exp\left[-\frac{i}{\hbar}\frac{\theta}{2}(\hat{x}\hat{p}_y - \hat{y}\hat{p}_x)\right], \tag{56}$$

$$\hat{V}_3(t) = \exp -\frac{i}{4\hbar}\left(\dot{m}_1(t)\hat{x}^2 + \dot{m}_2(t)\hat{y}^2\right). \tag{57}$$

Substituting equation (54) in equation (53), we can obtain a transformed Hamiltonian that is merely the coupling of two harmonic oscillators having frequencies $\Omega_1(t)$ and $\Omega_2(t)$ and unit masses:

$$\begin{aligned}\hat{H}_3(\hat{x}, \hat{y}, t) &= \hat{V}^{-1}(t)\hat{H}_2(\hat{x}, \hat{y}, t)\hat{V}(t) - i\hbar\hat{V}^{-1}(t)\frac{\partial}{\partial t}\hat{V}(t) \\ &= \frac{1}{2}\left(\hat{p}_x^2 + \hat{p}_y^2\right) + \frac{1}{2}\Omega_1^2(t)\hat{x}^2 + \frac{1}{2}\Omega_2^2(t)\hat{y}^2. \end{aligned}\tag{58}$$

At this stage, one can easily confirm that the relation given in equation (40) is correct, since it is consistent with equation (58). From unitary operators (44), (49), (50), (55) and (56), we can confirm that $\hat{\Lambda}_1(t)$ and $\hat{V}_1(t)$ are the squeeze operators whereas $\hat{U}(t)$, $\hat{\Lambda}_2(t)$ and $\hat{V}_2(t)$ are the rotation operators with the angles $\phi(t)$, $\frac{\pi}{4}$ and $\frac{\theta}{2}$, respectively.

Though the original Hamiltonian (41) involves the static coupling term xy and the dynamic coupling term $p_x p_y$, the transformed Hamiltonian (58) does not have such terms. Hence we can easily handle equation (58). In the following section, we establish the quantum solution (wave function) in the transformed system. And then, we will take the advantage of the unitary transformation starting form this wave function using the same unitary operators introduced in this section in order to derive the full wave functions in the original system.

5. QUANTUM SOLUTIONS

The problem of the harmonic oscillator with time-dependent mass and frequency can be transformed to that of the harmonic oscillator via the associated invariant [3]. It is easy to verify from Liouville-Von Neumann equation

$$\frac{d\hat{I}}{dt} = \frac{\partial \hat{I}}{\partial t} + \frac{1}{i\hbar}[\hat{I}, \hat{H}_3] = 0, \tag{59}$$

that the invariant associated with the transformed Hamiltonian of two-dimensional harmonic oscillator is given by

$$\begin{aligned} \hat{I}(\hat{x}, \hat{y}, t) &= \hat{I}(\hat{x}, t) + \hat{I}(\hat{y}, t) \\ &= \frac{1}{2}\left[\left(\frac{\hat{x}}{\rho_1}\right)^2 + \left(\rho_1 \dot{\hat{x}} - \dot{\rho}_1 \hat{x}\right)^2\right] + \frac{1}{2}\left[\left(\frac{\hat{y}}{\rho_2}\right)^2 + \left(\rho_2 \dot{\hat{y}} - \dot{\rho}_2 \hat{y}\right)^2\right], \end{aligned} \tag{60}$$

where $\rho_1(t)$ and $\rho_2(t)$ are c-number quantity satisfying the auxiliary equations

$$\ddot{\rho}_1 + \Omega_1^2(t)\rho_1 = 1/\rho_1^3, \tag{61}$$

$$\ddot{\rho}_2 + \Omega_2^2(t)\rho_2 = 1/\rho_2^3. \tag{62}$$

In order to make the invariant hermitian, $\hat{I}^\dagger = \hat{I}$, we choose only the real solution of (61) and (62). Then, we can find the eigenfunctions $\xi_{n_1 n_2}(x, y, t)$ of $\hat{I}(\hat{x}, \hat{y}, t)$, which are a complete orthonormal set corresponding to the time-independent eigenvalues $\lambda_{n_1 n_2}$, from the eigenvalue equation

$$\hat{I}(\hat{x}, \hat{y}, t)\xi_{n_1 n_2}(x, y, t) = \lambda_{n_1 n_2}\xi_{n_1 n_2}(x, y, t), \tag{63}$$

where

$$\lambda_{n_1n_2}=\hbar\left(n_1+\frac{1}{2}\right)+\hbar\left(n_2+\frac{1}{2}\right). \tag{64}$$

By evaluating equation (63) with the use of equation (60), we have the eigenstates in the form

$$\begin{aligned}\xi_{n_1n_2}(x,y,t) \;&=\; \left[\frac{1}{\pi\hbar n_1!n_2!2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2}\times H_{n_1}\left(\frac{x}{\hbar^{1/2}\rho_1}\right)H_{n_2}\left(\frac{y}{\hbar^{1/2}\rho_2}\right)\\ &\times\exp\left[\frac{i}{2\hbar}\left(\frac{\dot\rho_1}{\rho_1}+\frac{i}{\rho_1^2}\right)x^2+\frac{i}{2\hbar}\left(\frac{\dot\rho_2}{\rho_2}+\frac{i}{\rho_2^2}\right)y^2\right], \end{aligned}\tag{65}$$

where H_{n_1} and H_{n_2} are the usual Hermite polynomial of order n_1 and n_2, respectively.

The solution of the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\chi_{n_1n_2}(x,y,t)=\hat H_3(\hat x,\hat y,t)\chi_{n_1n_2}(x,y,t), \tag{66}$$

can be written in the form

$$\chi_{n_1n_2}(x,y,t)=e^{i\alpha_{n_1n_2}(t)}\xi_{n_1n_2}(x,y,t), \tag{67}$$

where the phase function $\alpha_{n_1n_2}(t)$ satisfy the equation

$$\frac{\partial}{\partial t}\alpha_{n_1n_2}(t)=\frac{1}{\hbar}\left\langle\xi_{n_1n_2}(x,y,t)\right|\frac{\partial}{\partial t}-\hat H_3(\hat x.\hat y,t)\left|\xi_{n_1n_2}(x,y,t)\right\rangle. \tag{68}$$

According to equations (67) and (68), the solutions $\chi_{n_1n_2}(x,y,t)$ of the transformed Schrödinger equation (66) are given by

$$\begin{aligned}\chi_{n_1n_2}(x,y,t) \;&=\; e^{i\alpha_{n_1n_2}(t)}\left[\frac{1}{\pi\hbar n_1!n_2!2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2}\times H_{n_1}\left(\frac{x}{\hbar^{1/2}\rho_1}\right)H_{n_2}\left(\frac{y}{\hbar^{1/2}\rho_2}\right)\\ &\times\exp\left[\frac{i}{2\hbar}\left(\frac{\dot\rho_1}{\rho_1}+\frac{i}{\rho_1^2}\right)x^2+\frac{i}{2\hbar}\left(\frac{\dot\rho_2}{\rho_2}+\frac{i}{\rho_2^2}\right)y^2\right], \end{aligned}\tag{69}$$

where the phase functions takes the form

$$\alpha_{n_1n_2}(t)=-\left(n_1+\frac{1}{2}\right)\int_0^t\frac{dt'}{\rho_1^2(t')}-\left(n_2+\frac{1}{2}\right)\int_0^t\frac{dt'}{\rho_2^2(t')}. \tag{70}$$

The relation between the wave functions, $\Psi_{n_1n_2}(x,y,t)$ in original system described by the Hamiltonian (41) and the wave functions $\chi_{n_1n_2}(x,y,t)$ in the transformed system is

$$\begin{aligned}\Psi_{n_1n_2}(x,y,t) \;&=\; \hat U(t)\hat\Lambda(t)\hat V(t)\chi_{n_1n_2}(x,y,t)\\ &=\; \hat U(t)\hat\Lambda_1(t)\hat\Lambda_2(t)\hat V_1(t)\hat V_2(t)\hat V_3(t)\chi_{n_1n_2}(x,y,t). \end{aligned}\tag{71}$$

Using equations (69), (70) and (71), we derive the full wave functions in the form

$$\begin{aligned}
\Psi_{n_1 n_2}(x,y,t) \; &= \; \left[\frac{(m_1(t)m_2(t))^{1/2}}{\pi\hbar n_1! n_2! 2^{n_1+n_2}\rho_1\rho_2}\right]^{1/2} \\
&\times H_{n_1}\left(\frac{(\eta_1\cos\phi+\eta_2\sin\phi)\,x+(-\eta_1\sin\phi+\eta_2\cos\phi)\,y}{\hbar^{1/2}\rho_1}\right) \\
&\times H_{n_2}\left(\frac{(\mu_1\cos\phi+\mu_2\sin\phi)\,x+(-\mu_1\sin\phi+\mu_2\cos\phi)\,y}{\hbar^{1/2}\rho_2}\right) \\
&\times \exp\frac{i}{2\hbar}\left[\left(\frac{m_-}{m_+}\right)^{1/2} f_1\cos^2\phi+\left(\frac{m_+}{m_-}\right)^{1/2} f_2\sin^2\phi+\frac{f_3}{2}\sin 2\phi\right]x^2 \\
&\times \exp\frac{i}{2\hbar}\left[\left(\frac{m_-}{m_+}\right)^{1/2} f_1\cos^2\phi+\left(\frac{m_+}{m_-}\right)^{1/2} f_2\sin^2\phi-\frac{f_3}{2}\sin 2\phi\right]y^2 \\
&\times \exp\frac{i}{2\hbar}\left[\left(\left(\frac{m_+}{m_-}\right)^{1/2} f_2-\left(\frac{m_-}{m_-}\right)^{1/2} f_1\right)\sin 2\phi+f_3\cos 2\phi\right]xy \\
&\times \exp i\left[-\left(n_1+\frac{1}{2}\right)\int_0^t\frac{dt'}{\rho_1^2(t')}-\left(n_2+\frac{1}{2}\right)\int_0^t\frac{dt'}{\rho_2^2(t')}\right], \qquad (72)
\end{aligned}$$

where the time-dependent coefficients $f_1(t)$, $f_2(t)$, $f_3(t)$, $\gamma(t)$, $\beta(t)$, $\eta_1(t)$, $\eta_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ are given as follows

$$\begin{aligned}
f_1(t) \; &= \; \left(\frac{\gamma}{2}m_1+\frac{\beta}{2}m_2\right)\cos^2\theta/2+\left(\frac{\gamma}{2}m_2+\frac{\beta}{2}m_1\right)\sin^2\theta/2 \\
&+\sqrt{m_1 m_2}\,(\beta-\gamma)\sin\theta/2\cos\theta/2, \qquad (73)
\end{aligned}$$

$$\begin{aligned}
f_2(t) \; &= \; \left(\frac{\gamma}{2}m_1+\frac{\beta}{2}m_2\right)\cos^2\theta/2+\left(\frac{\gamma}{2}m_2+\frac{\beta}{2}m_1\right)\sin^2\theta/2 \\
&-\sqrt{m_1 m_2}\,(\beta-\gamma)\sin\theta/2\cos\theta/2, \qquad (74)
\end{aligned}$$

$$f_3(t) = (-\gamma m_1+\beta m_2)\cos^2\theta/2+(\gamma m_2-\beta m_1)\sin^2\theta/2, \qquad (75)$$

$$\gamma(t) = \left(\frac{\dot\rho_1}{\rho_1}+\frac{i}{\rho_1^2}-\frac{\dot m_1(t)}{2}\right), \qquad (76)$$

$$\beta(t) = \left(\frac{\dot\rho_2}{\rho_2}+\frac{i}{\rho_2^2}-\frac{\dot m_2(t)}{2}\right), \qquad (77)$$

$$\eta_1(t) = \left(\frac{m_-}{m_+}\right)^{1/4}\left(\sqrt{m_1/2}\cos\theta/2-\sqrt{m_2/2}\sin\theta/2\right), \qquad (78)$$

$$\eta_2(t) = \left(\frac{m_+}{m_-}\right)^{1/4} \left(-\sqrt{m_1/2}\cos\theta/2 - \sqrt{m_2/2}\sin\theta/2\right), \tag{79}$$

$$\mu_1(t) = \left(\frac{m_-}{m_+}\right)^{1/4} \left(\sqrt{m_1/2}\sin\theta/2 + \sqrt{m_2/2}\cos\theta/2\right), \tag{80}$$

$$\mu_2(t) = \left(\frac{m_+}{m_-}\right)^{1/4} \left(-\sqrt{m_1/2}\sin\theta/2 + \sqrt{m_2/2}\cos\theta/2\right). \tag{81}$$

The final solutions given in (72) are somewhat complicate, but they are very useful when predicting the evolution of the probability distribution of the system. A considerable physical significance of such analytical solutions is that their application in physical system is very flexible, even when the internal and external situations of the system vary from time to time. Numerical solutions obtained from, for example, the FDTD (finite difference time domain)[46] method are however inconvenient as inputs to further analyses, especially when the parameters of the charged particle system vary with time like in this case. One can easily take a complete description of how the charged particle behaves under the given Hamiltonian, by means of this analytical wave function.

6. CONCLUSION

Though the motion of charged particles in magnetic fields is a fascinating problem in both quantum and classical view, most of the relevant research is concentrated on static problem that can be described by time-*in*dependent Hamiltonian. In this paper, this problem is generalized to a more complicated case that is described in terms of time-dependent Hamiltonian, by supposing that the parameters such as the effective mass of the charged particle vary explicitly with time in the presence of variable magnetic field. We have presented an alternative treatment after reducing the problem related to the charged particle motion to that of the quantal time-dependent coupled oscillators.

We approached the problem in two ways. In one, we used a time-dependent generalized canonical transformations which enabled us to transform the initial classical Hamiltonian (2) to a more simplified one associated with the two harmonic oscillators having time-dependent

frequencies $\Omega_1(t)$ and $\Omega_2(t)$. We have taken, as an another way, an alternative approach based on unitary transformation that allowed us to transform the quantal Hamiltonian (41) to an equally simple one, but in the framework of quantum mechanics. To facilitate the derivation of quantum states, we introduced dynamical invariant operator $\hat{I}(\hat{x}, \hat{y}, t)$. The eigenvalues and eigenstates of the invariant operator are obtained using the Liouville-Von Neumann equation.

We derived the exact wave functions of the system on the basis of the fact that they are the same as the eigenstates of the invariant operator, except for some time-dependent phase factors $\exp\left[i\alpha_{n_1 n_2(t)}\right]$. The wave functions in the transformed system presented in equation (69) are relatively simple and expressed in terms of Hermite polynomial. However, as you can see from equation (72) with equations (73)-(81), the wave functions in the original system are somewhat complicate. These wave functions are represented in terms of ρ_1 and ρ_2 that are the solutions of some classical equation of motion given in Eqs. (61) and (62), respectively. If we let $x_{c,1}$ and $x_{c,2}$ as the two independent classical solutions of the x-component transformed Hamiltonian presented in (58), ρ_1 can be written as $\rho_1 = [a_1 x_{c,1}^2 + a_2 x_{c,1} x_{c,2} + x_{c,2}^2]^{1/2}$ (see Ref. [47] for rigorous mathematical proof). Of course, y-component two independent solutions $y_{c,1}$ and $y_{c,2}$ hold the similar relation with ρ_2: $\rho_2 = [b_1 y_{c,1}^2 + b_2 y_{c,1} y_{c,2} + b_3 y_{c,2}^2]^{1/2}$. This implies that we can take the complete knowledge for the behavior of the system within the scope that quantum mechanics admits when we exactly know the classical solutions of the *transformed system*, since we did not used any approxination in the development of our theory.

Though we expressed the wave functions in terms of the classical solutions of *transformed system*, many authors represent the wave functions for time-dependent Hamiltonian system in terms of the classical solutions of *original system*. However, in this case, it is unclear that we can represent the wave functions using the classical solutions of *original system* since the classical solutions in the original system can not be decoupled into each coordinate component due to the existence of coupled terms $\hat{x}\hat{y}$ and $\hat{p}_x\hat{p}_y$ in equation (41). Even if it were possible to manage the problem using those of *original system*, the corresponding mathematical procedure to obtain the exact wave functions would be very difficult and the results would become much more complicate. The wave functions we obtained here can be used to evaluate not only the

quantum mechanical expectation values of various observables such as physical momentum and quantum energy but also probability densities and fluctuations of the canonical variables.

Finally, we did not considered thermal effects in this work. The quantum behaviors of the charged particle motion with consideration of thermal effects may be a good topic as a next task.

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